コンパクト単純リー群上の不変なアインシュタイン計量について

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On invariant Einstein metrics on compact simple Lie groups

based on joint works with
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- introduction
- Naturally reductive metrics, results of D'Atri and Ziller
- Ricci tensor of a compact homogeneous space
- Known results on Non-naturally reductive Einstein metrics on compact Lie groups
- Case of compact Lie groups $SU(N)$

(M, g): Riemannian manifold

- (M, g) is called Einstein if the Ricci tensor $\mathrm{r}(g)$ of the metirc g satisfies $\mathrm{r}(g) = c\, g$ for some constant c.

  We consider G-invariant Einstein metrics on a homogeneous space G/H.

- General Problem: Find G-invariant Einstein metrics on a homogeneous space G/H and classify them if it is not unique.

- Einstein homogeneous spaces can be diveded into three cases depending on Einstein constant c.
  Here we consider the case $c > 0$.

Examples of Einstein manifolds
(we see that G/H is compact and $\pi_1(G/H)$ is finite)

- Sphere $(S^n = \mathrm{SO}(n+1)/\mathrm{SO}(n), g_0)$,
- Complex Projective space
  $(\mathbb{C}P^n = \mathrm{SU}(n+1)/(\mathrm{S}(\mathrm{U}(1) \times \mathrm{U(n)}), g_0)$,
- Symmetric spaces of compact type,
  isotropy irreducible spaces (in these cases G-invariant Einstein metrics is unique up to a constant multiple)
- In particular, compact semi-simple Lie group with a bi-inavariant metric
- Generalized flag manifolds (Kähler C-spaces) (which admit Kähler Einstein metrics)
  (G-invariant Einstein metrics may not be unique as real manifold.)

Introduction (non-existence and existence)

- (Wang-Ziller 1986) There exist compact homogeneous spaces G/H **with no G-invariant Einstein metrics.**
- For example, let $G = \mathrm{SU}(4)$, $K = \mathrm{Sp}(2)$, $H = \mathrm{SU}(2)$ (where $\mathrm{SU}(2)$ is a maxmal subgroup of $\mathrm{Sp}(2)$).
  Then G/H has no (G-)invariant Einstein metrics. Note that $\dim G/H = 12$.
- How about the case that $\dim G/H < 12$?
- (Böhm-Kerr (2006)) For a simply connected compact homogeneous space G/H of $\dim G/H \leq 11$, there exists always a G-invariant Einstein metric on G/H.

- Known results on small dimensions

  (Nikonorov, Rodionov (2003)) For a simply connected compact homogeneous space G/H of $\dim G/H \leq 7$, all G-invariant Einstein metrics has been determined on G/H, except $\mathrm{SU}(2) \times \mathrm{SU}(2)$.

- (Wang-Ziller (1990))
  There are infinitely many principal S^1-bundles over $\mathbb{C}P^1 \times \mathbb{C}P^1$ which are all diffeomorphic to $S^2 \times S^3$,
  but as homogeneous spaces $(\mathrm{SU}(2) \times \mathrm{SU}(2))/S^1$, they are quite different.
  In fact, we can see that the moduli space of Einstein metrics on $S^2 \times S^3$ has infinitely many components by using these Einstein metrics.

Compact Lie groups (case $c > 0$)

- A compact simple Lie group with a bi-invariant metric (for example given by negative of Killing form) is Einstein.
- In 1971, Jensen obtained Einstein metric on a compact simple Lie group G which is not bi-invariant, except those locally isomorphic to $\mathrm{SO}(3)$, G_2 and $\mathrm{Sp}(2n+1)$.
- In 1979, D'Atri and Ziller obtained many Einstein metrics on a compact Lie group G, which are naturally reductive.
- Open problem: Find all left-invariant Einstein metrics on a compact simple Lie group G.
  How many are there? (finite or infinite)
- Even for $G = \mathrm{SU}(3)$, or $G = \mathrm{SU}(2) \times \mathrm{SU}(2)$, we do not know all left-invariant Einstein metrics on G. (finite or infinite)

- It is known that there exist at least two left invariant Einstein metrics on $\mathrm{SU}(2) \times \mathrm{SU}(2)$. One of these metrics is standard, the second metric ρ_J was found by G. Jensen.

- Nikonorov and Rodionov (2003) has computed the scalar curvature of left invariant metrics on $\mathrm{SU}(2) \times \mathrm{SU}(2)$. There is 14-parameters for the metrics and it seems to be difficult to obtain critical points (Einstein metrics).

- Theorem (Nikonorov and Rodionov (2003)).
  Let g be a left-invariant Einstein metric on the Lie group $\mathrm{SU}(2) \times \mathrm{SU}(2)$ which is Ad(S^1)-invariant with respect to a certain embedding $S^1 \subset \mathrm{SU}(2) \times \mathrm{SU}(2)$. Then the metric g is isometric (up to a homothety) to one of the metrics above.

Naturally reductive metrics

- (M, g) : a compact Riemannian manifold
  $I(M, g)$: the Lie group of all isometries of M (compact)

  A Riemannian manifold (M, g) is K-homogeneous if a closed subgroup K of $I(M, g)$ acts transitively on M.

  For a K-homogeneous Riemannian manifold (M, g), we write $M = K/L$, where L is the isotropy subgroup of K at a point o.

- $\mathfrak{k}$: the Lie algebra of K
  $\mathfrak{l}$: the subalgebra corresponding to L
  $\mathfrak{p}$: a complement subspace of $\mathfrak{k}$ to $\mathfrak{l}$ with $\mathrm{Ad}(L)\mathfrak{p} \subset \mathfrak{p}$
  $\mathfrak{k} = \mathfrak{l} \oplus \mathfrak{p}$

  Pull back the inner product g_o on $T_o(M)$ to an inner product on $\mathfrak{p}$, denoted by $< \, , \, >$.
  $< \, , \, >$ is an $\mathrm{Ad}(L)$-invariant inner product on $\mathfrak{p}$

Naturally reductive metrics

- For $X \in \mathfrak{k}$, we will denote by $X_{\mathfrak{l}}$ (resp. $X_{\mathfrak{p}}$) the $\mathfrak{l}$-component (resp. $\mathfrak{p}$-component) of X.
- A homogeneous Riemannian metric on M is said to be naturally reductive with respect to K, if there exist K and $\mathfrak{p}$ as above such that

$$< [Z, X]_{\mathfrak{p}} , Y > + < X, [Z, Y]_{\mathfrak{p}} >= 0 \quad \text{for} \quad X, Y, Z \in \mathfrak{p}.$$

That is, when we write the Riemannain connection ∇ as, for $X, Y \in \mathfrak{p}$,

$$\nabla_X Y = -\frac{1}{2}[X, Y]_{\mathfrak{p}} + U(X, Y),$$

$U(X, Y) = 0$ for any $X, Y \in \mathfrak{p}$.

Naturally reductive metrics on a compact Lie group

- D'Atri and Ziller (Memoirs Amer. Math. Soc. 19 (215) (1979)) investigated naturally reductive metrics among the left invariant metrics on compact Lie groups and obtained a complete classification of the metrics in the case of simple Lie groups.
- For a compact semi-simple Lie group G and a closed subgroup H, the group $G \times H$ acts transitively on G by

$$(g, h)y = gyh^{-1} \quad ((g, h) \in G \times H,\ y \in G)$$

  and the Lie group G can be expressed as $(G \times H)/\Delta H$, where $\Delta H = \{(h, h) \mid h \in H\}$.
- Note that the Killing form of a compact semi-simple Lie algebra $\mathfrak{g}$ is negative definite. We set $B = -$ Killing form. Then B is an Ad(G)-invariant inner product on $\mathfrak{g}$.

- Let $\mathfrak{m}$ be an orthogonal complement of $\mathfrak{h}$ (the Lie algebra of the Lie subgroup H) in $\mathfrak{g}$ with respect to B. Then we have

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m}, \quad \mathrm{Ad}(H)\mathfrak{m} \subset \mathfrak{m}.$$

- Let $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_p$ be the decomposition into ideals of $\mathfrak{h}$, where $\mathfrak{h}_0$ is the center of $\mathfrak{h}$ and $\mathfrak{h}_i$ $(i = 1, \cdots, p)$ are simple ideals of $\mathfrak{h}$. Let $A_0|_{\mathfrak{h}_0}$ be an arbitrary metric on $\mathfrak{h}_0$.

Theorem

(D'Atri-Ziller 1979) *Under the notations above, a left invariant metric* $<\ ,\ >$ *on* G *of the form*

$$<\ ,\ >= x \cdot B|_{\mathfrak{m}} + A_0|_{\mathfrak{h}_0} + u_1 \cdot B|_{\mathfrak{h}_1} + \cdots + u_p \cdot B|_{\mathfrak{h}_p} \qquad (x, u_1, \cdots, u_p \in \mathbb{R}_+) \tag{1}$$

is naturally reductive with respect to $G \times H$*.*
Note that $(G = (G \times H)/\Delta H)$ *.*

Conversely, if a left invariant metric $<\ ,\ >$ *on a compact simple Lie group* G *is naturally reductive, then there exists a closed subgroup* H *of* G *and the metric* $<\ ,\ >$ *is given of the form* (1).

- D'Atri and Ziller (1979) have investigated **naturally reductive Einstein metrics** on a compact simple Lie group G in the case which Ad(H) acts on $\mathfrak{m}$ irreducibly, which includes the left invariant metric determined by **irreducible symmetric spaces of compact type and isotropy irreducible spaces**.
- In particular, D'Atri and Ziller found at least the following number of left invariant Einstein metrics:
- $n+1$ on $\mathrm{SU}(2n+2),\ \mathrm{SU}(2n+3),\ \mathrm{Sp}(2n),\ \mathrm{Sp}(2n+1)$,
- $3n-2$ on $\mathrm{SO}(2n),\ \mathrm{SO}(2n+1)$, $(n \geq 3)$
- for exceptional Lie groups,
  5 on G_2, 10 on F_4, 14 on E_6, 15 on E_7, 11 on E_8.

Invariant metrics on a compact Lie group

- D'Atri and Ziller (1979) asked a following question:
  Is there non-naturally reductive left invariant Einstein metrics on a compact Lie group?
- First we consider the case when $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_q$ is a decomposition into irreducible Ad(H)-modules $\mathfrak{m}_j$ $(j = 1, \cdots, q)$ and Ad(H)-modules $\mathfrak{m}_j$ are **mutually non-equivalent** and $\mathbf{dim}\, \mathfrak{h}_0 \leq \mathbf{1}$.
- We consider the following left invariant metric on G which is Ad(H)-invariant:
  $$< \, , \, >= u_0 B|_{\mathfrak{h}_0} + u_1 B|_{\mathfrak{h}_1} + \cdots + u_p B|_{\mathfrak{h}_p} + x_1 B|_{\mathfrak{m}_1} + \cdots + x_q B|_{\mathfrak{m}_q} \quad (2)$$
  where $u_0, u_1, \cdots, u_p, x_1, \cdots, x_q \in \mathbb{R}_+$, and the G-invariant Riemannian metric on G/H:
  $$(\, , \,) = x_1 B|_{\mathfrak{m}_1} + \cdots + x_q B|_{\mathfrak{m}_q}. \quad (3)$$

Invariant metrics on a compact Lie group

- Note that left invariant symmetric covariant 2-tensors on G which are $\mathrm{Ad}(H)$-invariant are the same form as the metrics, and this is also true for G-invariant symmetric covariant 2-tensors on G/H.
- In particular, the Ricci tensor r of a left invariant Riemannian metric $< \ , \ >$ on G is a left invariant symmetric covariant 2-tensor on G which is $\mathrm{Ad}(H)$-invariant and thus r is of the same form as (2), and Ricci tensor $\bar{r}$ of a G-invariant Riemannian metric on G/H is of the same form as (3).
- For simplicity, we write the decomposition $\mathfrak{g} = \mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_p \oplus \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_q$ (resp. $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_q$) as $\mathfrak{g} = \mathfrak{w}_0 \oplus \mathfrak{w}_1 \oplus \cdots \oplus \mathfrak{w}_p \oplus \mathfrak{w}_{p+1} \oplus \cdots \oplus \mathfrak{w}_{p+q}$ (resp. $\mathfrak{m} = \mathfrak{w}_{p+1} \oplus \cdots \oplus \mathfrak{w}_{p+q}$).

Invariant metrics on a compact Lie group

- For simplicity, we now write a metric of the form (2) on a compact Lie group G as follows:

$$g = y_0 \cdot B|_{\mathfrak{w}_0} + y_1 \cdot B|_{\mathfrak{w}_1} + \cdots + y_p \cdot B|_{\mathfrak{w}_p} + y_{p+1} \cdot B|_{\mathfrak{w}_{p+1}} + \cdots + y_{p+q} \cdot B|_{\mathfrak{w}_{p+q}} \tag{4}$$

  and a metric of the form (3) on a compact space G/H as follows:

$$h = w_{p+1} \cdot B|_{\mathfrak{w}_{p+1}} + \cdots + w_{p+q} \cdot B|_{\mathfrak{w}_{p+q}} \tag{5}$$

- Note that the metric of the form (4) is naturally reductive on a compact simple Lie group G with respect to $G \times H$ if and only if $y_{p+1} = \cdots = y_{p+q}$.

- Let $\{e_\alpha\}$ be a B-orthonormal basis adapted to the decomposition of $\mathfrak{g}$, i.e., $e_\alpha \in \mathfrak{w}_i$ for some i, and $\alpha < \beta$ if $i < j$ (with $e_\alpha \in \mathfrak{w}_i$ and $e_\beta \in \mathfrak{w}_j$).
- We put $A^\gamma_{\alpha\beta} = B(\left[e_\alpha, e_\beta\right], e_\gamma)$, so that $\left[e_\alpha, e_\beta\right] = \sum_\gamma A^\gamma_{\alpha\beta} e_\gamma$, and set $\begin{bmatrix} k \\ ij \end{bmatrix} = \sum (A^\gamma_{\alpha\beta})^2$, where the sum is taken over all indices α, β, γ with $e_\alpha \in \mathfrak{w}_i,\ e_\beta \in \mathfrak{w}_j,\ e_\gamma \in \mathfrak{w}_k$. Then, $\begin{bmatrix} k \\ ij \end{bmatrix}$ is independent of the B-orthonormal bases chosen for $\mathfrak{w}_i, \mathfrak{w}_j, \mathfrak{w}_k$, and

$$\begin{bmatrix} k \\ ij \end{bmatrix} = \begin{bmatrix} k \\ ji \end{bmatrix} = \begin{bmatrix} j \\ ki \end{bmatrix}. \tag{6}$$

The notations $\begin{bmatrix} k \\ ij \end{bmatrix}$ was introduced by Wang and Ziller to study Einstein metrics on compact homogeneous manifolds.

Lemma

Let $d_k = \dim \mathfrak{w}_k$*.*
(i) *The components* $r_0, r_1, \cdots, r_{p+q}$ *of Ricci tensor* r *of the metric* g *of the form* (2) *on* G *are given by*

$$r_k = \frac{1}{2y_k} + \frac{1}{4d_k}\sum_{j,i}\frac{y_k}{y_j y_i}\begin{bmatrix} k \\ ji \end{bmatrix} - \frac{1}{2d_k}\sum_{j,i}\frac{y_j}{y_k y_i}\begin{bmatrix} j \\ ki \end{bmatrix} \quad (k = 0, 1, \cdots, p+q),$$

where the sum is taken over $i, j = 0, 1, \cdots, p+q$*. Moreover, for each* k*, we have* $\sum_{i,j}\begin{bmatrix} j \\ ki \end{bmatrix} = d_k$*.*

Remark. If the dimension of the center $\mathfrak{h}_0$ is greater than or equal 2, we have to consider “off diagonal” part of Ricci tensor for $\dim \mathfrak{h}_0$.

Lemma

(ii) *The components* $\bar{r}_{p+1}, \cdots, \bar{r}_{p+q}$ *of Ricci tensor* $\bar{r}$ *of the metric* h *of the form* (3) *on* G/H *are given by*

$$\bar{r}_k = \frac{1}{2w_k} + \frac{1}{4d_k} \sum_{j,i} \frac{w_k}{w_j w_i} \begin{bmatrix} k \\ ji \end{bmatrix} - \frac{1}{2d_k} \sum_{j,i} \frac{w_j}{w_k w_i} \begin{bmatrix} j \\ ki \end{bmatrix} \quad (k = p+1, \cdots, p+q),$$

where the sum is taken over $i, j = p+1, \cdots, p+q$.

Known results on Non-naturally reductive Einstein metrics on compact Lie groups

- **Theorem**[K. Mori, 1996]
  On a compact Lie group $\mathrm{SU}(n)$ $(n \geq 6)$, there exist non-naturally reductive Einstein metrics. (preprint) (Generalized flag manifolds and/or Generalized Wallach spaces)

  For this case, the space of the metrics has been studied from $\mathrm{SU}(2+2+n-4)/\,\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(2)\times\mathrm{U}(n-4))$ (≥ 6). (Note that $\dim \mathfrak{h}_0 = 2$ in these cases.)

- **Theorem**[Arvanitoyeorgos, Mori and S., 2008] (Geom. Dedicata)
  On a compact simple Lie group G, either $\mathrm{SO}(n)$ $(n \geq 11)$, $\mathrm{Sp}(n)$ ($n \geq 3$), E_6, E_7 or E_8, there exist non-naturally reductive Einstein Einstein metrics. (Generalized flag manifolds with two irreducible summands)

- **Theorem**[Chen and Liang, 2014] (Ann. Glob. Anal. Geom.)
  On the compact Lie group F_4 there exists a non-naturally reductive Einstein Einstein metric.
  (Generalized Wallach spaces $F_4/\,\mathrm{SO}(8)$ with fiber $\mathrm{SO}(9)/\,\mathrm{SO}(8)$)

- **Theorem**[Arvanitoyeorgos, S. and Statha, 2015]
  (Geometry, Imaging and Computing vol. 2.2)
  The compact simple Lie groups $\mathrm{SO}(n)$ $(n \geq 7)$ admit left-invariant Einstein metrics which are not naturally reductive.
  (Generalized Wallach spaces
  $\mathrm{SO}(3+3+n-6)/\,\mathrm{SO}(3)\times\mathrm{SO}(3)\times\mathrm{SO}(n-6)$)

- **Theorem**[Arvanitoyeorgos, S. and Statha]
  (Proceedings of ICDG2014 , 2015)

  The compact simple Lie groups $\mathrm{Sp}(n)$ $(n \geq 3)$ admit left-invariant Einstein metrics which are not naturally reductive.

  (Generalized Wallach spaces $\mathrm{Sp}(n)/\mathrm{Sp}(n-2) \times \mathrm{Sp}(1) \times \mathrm{Sp}(1)$)

- **Theorem**[Chrysikos and S.]
  (to appear J. of Geom. Phy. 116 (2017)) (arXiv:1511.03993)

  The compact simple Lie groups G_2, F_4, E_6, E_7 and E_8 admit left-invariant Einstein metrics which are not naturally reductive.

  (Generalized flag manifolds G/H with the second Betti number $b_2(G/H) = 1$ and three irreducible summands)

- **Theorem**[Arvanitoyeorgos, S. and Statha]

  On a compact Lie group $\mathrm{SU}(n+3)$ $(n \geq 2)$,
  there exist non-naturally reductive Einstein metrics which are different from K. Mori's results.

  The space of the metrics has been studied from
  $\mathrm{SU}(1+2+n)/\,\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(2) \times \mathrm{U}(n))$ for $n \geq 2$.
  (Generalized flag manifolds and/or Generalized Wallach spaces)
  Note that in this case The Lie algebra of the group
  $\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(2) \times \mathrm{U}(n))$ has two dimensional center.

- **Theorem**[Arvanitoyeorgos, S. and Statha]
  (To appear Proceedings of ICDG2016 , 2017)

  On a compact Lie group $\mathrm{SU}(n+3)$ $(n \geq 5)$, there exist non-naturally reductive Einstein metrics which are different from K. Mori's results.

  The space of the metrics has been studied from $\mathrm{SU}(3+n)/(\mathrm{U}(1)\,\mathrm{SO}(3)\,\mathrm{SU}(n))$ for $n \geq 5$, where the Lie subgroup $\mathrm{SO}(3)$ is a natural subgroup of $\mathrm{SU}(3)$ and $\mathrm{SU}(3+n)/(\mathrm{S}(\mathrm{U}(3) \times \mathrm{U}(n))$ is a complex Grassmann manifold.

Summary for compact simple Lie groups

- Now we want to summarize the results for left-invariant non-naturally reductive Einstein metrics on compact simple Lie groups.
- For $\mathrm{SU}(n)$ $(n \geq 5)$, $\mathrm{SO}(n)$ ($n \geq 7$), $\mathrm{Sp}(n)$ ($n \geq 3$), E_6, E_7, E_8, F_4 and G_2,
  there exist non-naturally reductive Einstein metrics.
  (Recently we obtained more non-naturally reductive Einstein metrics on $\mathrm{SU}(n)$ $(n \geq 5$) which are different from K. Mori's results. In particular, the case for $\mathrm{SU}(5)$ is the first example.)
- For the cases of $\mathrm{SU}(n)$ $(n = 3, 4)$, we still do not know whether there exist non-naturally reductive Einstein metrics or not.
- $\mathrm{SO}(5) = \mathrm{Sp}(2)$ (locally) and $\mathrm{SO}(6) = \mathrm{SU}(4)$ (locally) are still open.

- We consider the homogeneous space $G/H = \mathrm{SU}(l+m+n)/\,\mathrm{S}(\mathrm{U}(l)\times\mathrm{U}(m)\times\mathrm{U}(n))$, which is a complex generalized flag manifold. The tangent space $\mathfrak{m}$ of G/H decomposes into three $\mathrm{Ad}(H)$-submodules

$$\mathfrak{m} = \mathfrak{m}_{12} \oplus \mathfrak{m}_{13} \oplus \mathfrak{m}_{23},$$

where

$$\mathfrak{m}_{12} = \left\{ \begin{pmatrix} 0 & A & 0 \\ -\bar{A}^t & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : A \in M(l,m) \right\}, \mathfrak{m}_{13} = \left\{ \begin{pmatrix} 0 & 0 & B \\ 0 & 0 & 0 \\ -\bar{B}^t & 0 & 0 \end{pmatrix} : B \in M(l,n) \right\},$$

$$\mathfrak{m}_{23} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & C \\ 0 & -\bar{C}^t & 0 \end{pmatrix} : C \in M(m,n) \right\}.$$

Note that the irreducible $\mathrm{Ad}(H)$-submodules $\mathfrak{m}_{12}$, $\mathfrak{m}_{13}$ and $\mathfrak{m}_{23}$ are mutually non-equivalent.

- Let $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \mathfrak{h}_3$ be the decomposition of $\mathfrak{h}$, the Lie algebra of H, into its center $\mathfrak{h}_0$, and simple ideals as follows:

$$\begin{aligned}
\mathfrak{h}_0 &= \left\{ \sqrt{-1} \begin{pmatrix} \frac{a}{l} I_l & 0 & 0 \\ 0 & \frac{b}{m} I_m & 0 \\ 0 & 0 & \frac{c}{n} I_n \end{pmatrix} : a+b+c=0, (a,b,c \in \mathbb{R}) \right\}, \\
\mathfrak{h}_1 &= \left\{ \begin{pmatrix} A_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : A_1 \in \mathfrak{su}(l) \right\}, \mathfrak{h}_2 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} : A_2 \in \mathfrak{su}(m) \right\}, \\
\mathfrak{h}_3 &= \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & A_3 \end{pmatrix} : A_3 \in \mathfrak{su}(n) \right\}.
\end{aligned}$$

The Lie algebra $\mathfrak{g} = \mathfrak{su}(l+m+n)$ splits into $\mathfrak{h}$ and three $\mathrm{Ad}(H)$-irreducible modules as follows:

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} = \mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \mathfrak{h}_3 \oplus \mathfrak{m}_{12} \oplus \mathfrak{m}_{13} \oplus \mathfrak{m}_{23}. \qquad (7)$$

This is an orthogonal decomposition with respect to B.
Let

$$H_4 = \begin{pmatrix} \frac{r}{l+m} I_l & 0 & 0 \\ 0 & \frac{r}{l+m} I_m & 0 \\ 0 & 0 & -\frac{r}{n} I_n \end{pmatrix} \text{ and } H_5 = \begin{pmatrix} \frac{s}{l} I_l & 0 & 0 \\ 0 & -\frac{s}{m} I_m & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then we have the B-orthogonal decomposition of $\mathfrak{h}_0$

$$\mathfrak{h}_0 = \mathfrak{h}_4 \oplus \mathfrak{h}_5,$$

where $\mathfrak{h}_4 = \mathrm{span}\{\sqrt{-1} H_4\}$, $\mathfrak{h}_5 = \mathrm{span}\{\sqrt{-1} H_5\}$.

We consider another basis $\{V_4, V_5\}$ of $\mathfrak{h}_0$ and let $\tilde{\mathfrak{h}}_4 = \mathrm{span}\{V_4\}$, $\tilde{\mathfrak{h}}_5 = \mathrm{span}\{V_5\}$. We relate the basis $\{V_4, V_5\}$ to the basis $\{H_4, H_5\}$ by

$$(H_4, H_5) = (V_4, V_5)\, P$$

where $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbb{R})$.

We define the inner product $\langle V_i, V_j \rangle = \delta_{ij}$ on $\mathfrak{h}_0 = \tilde{\mathfrak{h}}_4 \oplus \tilde{\mathfrak{h}}_5$. If the numbers a, b, c, d are viewed as parameters, then any inner product on $\mathfrak{h}_0$ are given by

$$\langle\langle \cdot, \cdot \rangle\rangle = u_4 \langle \cdot, \cdot \rangle|_{\tilde{\mathfrak{h}}_4} + u_5 \langle \cdot, \cdot \rangle|_{\tilde{\mathfrak{h}}_5}, \ \ u_4, u_5 > 0. \tag{8}$$

Thus the Lie algebra $\mathfrak{g}$ of $G = \mathrm{SU}(l+m+n)$ splits into direct sum of eight $\mathrm{Ad}(H)$-submodules as follows:

$$\mathfrak{g} = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \mathfrak{h}_3 \oplus \tilde{\mathfrak{h}}_4 \oplus \tilde{\mathfrak{h}}_5 \oplus \mathfrak{m}_{12} \oplus \mathfrak{m}_{13} \oplus \mathfrak{m}_{23}.$$

Now any left-invariant metric on $\mathrm{SU}(l+m+n)$ with $\mathrm{Ad}(H)$-invariant ($H = \mathrm{S}(\mathrm{U}(l)\times\mathrm{U}(m)\times\mathrm{U}(n))$)
is given by $\mathrm{Ad}(H)$-invariant inner product g on $\mathfrak{g} = \mathfrak{su}(l+m+n)$) of a form:

$$g = \langle\langle\cdot,\cdot\rangle\rangle|_{\mathfrak{h}_0} + u_1 B|_{\mathfrak{h}_1} + u_2 B|_{\mathfrak{h}_2} + u_3 B|_{\mathfrak{h}_3} + \sum_{1\le i<j\le 3} x_{ij} B|_{\mathfrak{m}_{ij}} \qquad (9)$$

To find non-naturally reductive Einstein metrics on $\mathrm{SU}(N)$ we need to determine the Ricci tensor for the metric g.

This will be done in two steps. The first step is to compute the Ricci tensor at the center $\mathfrak{h}_0$ and the second step is to compute the Ricci tensor for the diagonal part of metric g.

Observe that, in general, the inner product $\langle\langle\cdot,\cdot\rangle\rangle$ on $\mathfrak{h}_0$ is not $\mathrm{Ad}(G)$-invariant, that is, for $U_k \in \tilde{\mathfrak{h}}_k$ $(k = 4, 5)$,

$$\langle\langle [X, Y]\,,\, U_k\rangle\rangle \neq \langle\langle X\,,\, [Y, U_k]\rangle\rangle.$$

For inner product $Q = \langle\ ,\ \rangle|_{\tilde{\mathfrak{h}}_4} + \langle\ ,\ \rangle|_{\tilde{\mathfrak{h}}_5} + B|_{\mathfrak{h}_1} + B|_{\mathfrak{h}_2} + B|_{\mathfrak{h}_3} + \sum_{i<j} B|_{\mathfrak{m}_{ij}}$,

we define the numbers

$$\left\{ {k \atop ij} \right\} = \sum_{\alpha,\beta,k} Q([X_\alpha^{(i)}, X_\beta^{(j)}]\,,\, X_\gamma^{(k)})^2,$$

where $\{X_\alpha^{(i)}\}$ is a Q-orthonormal basis adapted to the decomposition of $\mathfrak{g}$. Note that, for $k = 4, 5$,

$$\left\{ {k \atop ij} \right\} = \left\{ {k \atop ji} \right\} \quad \text{and} \quad \left\{ {k \atop ij} \right\} \neq \left\{ {j \atop ik} \right\}.$$

Also we have:

$$\left\{\begin{matrix} 1 \\ 11 \end{matrix}\right\} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}, \ \left\{\begin{matrix} 2 \\ 22 \end{matrix}\right\} = \begin{bmatrix} 2 \\ 22 \end{bmatrix}, \ \left\{\begin{matrix} 3 \\ 33 \end{matrix}\right\} = \begin{bmatrix} 3 \\ 33 \end{bmatrix}, \ \left\{\begin{matrix} (12) \\ (13)(23) \end{matrix}\right\} = \begin{bmatrix} (12) \\ (13)(23) \end{bmatrix},$$

$\left\{\begin{matrix} j \\ ij \end{matrix}\right\} = \begin{bmatrix} j \\ ij \end{bmatrix}$, for $i = 1, 2, 3$ and $j = (12), (13), (23)$.

For the center $\mathfrak{h}_0$ we need to compute the following numbers:

$$\left\{\begin{matrix} 4 \\ (12)(12) \end{matrix}\right\}, \left\{\begin{matrix} 4 \\ (13)(13) \end{matrix}\right\}, \left\{\begin{matrix} 4 \\ (23)(23) \end{matrix}\right\}, \left\{\begin{matrix} 5 \\ (12)(12) \end{matrix}\right\}, \left\{\begin{matrix} 5 \\ (13)(13) \end{matrix}\right\}, \left\{\begin{matrix} 5 \\ (23)(23) \end{matrix}\right\}.$$

In fact, we see that

$$\begin{aligned}
\left\{ {4 \atop (12)(12)} \right\} &= \frac{b^2(\ell+m)}{\ell+m+n} \\
\left\{ {4 \atop (13)(13)} \right\} &= \frac{a^2\ell}{\ell+m} + \frac{b^2 mn}{(\ell+m+n)(\ell+m)} + \frac{2ab\sqrt{\ell mn}}{(\ell+m)\sqrt{(\ell+m+n)}} \\
\left\{ {4 \atop (23)(23)} \right\} &= \frac{a^2 m}{\ell+m} + \frac{b^2 \ell n}{(\ell+m+n)(\ell+m)} - \frac{2ab\sqrt{\ell mn}}{(\ell+m)\sqrt{(\ell+m+n)}} \\
\left\{ {5 \atop (12)(12)} \right\} &= \frac{d^2(\ell+m)}{\ell+m+n} \\
\left\{ {5 \atop (13)(13)} \right\} &= \frac{c^2\ell}{\ell+m} + \frac{d^2 mn}{(\ell+m+n)(\ell+m)} + \frac{2cd\sqrt{\ell mn}}{(\ell+m)\sqrt{(\ell+m+n)}} \\
\left\{ {5 \atop (23)(23)} \right\} &= \frac{c^2 m}{\ell+m} + \frac{d^2 \ell n}{(\ell+m+n)(\ell+m)} - \frac{2cd\sqrt{\ell mn}}{(\ell+m)\sqrt{(\ell+m+n)}}.
\end{aligned}$$

The Ricci tensor Ric_g of the left-invariant metric g on G is given as follows:

$$\begin{aligned}\mathrm{Ric}_g(X,Y) &= -\frac{1}{2}\sum_i \langle [X,X_i],[Y,X_i]\rangle + \frac{1}{2}B(X,Y) \\ &\quad +\frac{1}{4}\sum_{i,j}\langle [X_i,X_j],X\rangle\langle [X_i,X_j],Y\rangle. \end{aligned} \tag{10}$$

where $\{X_j\}$ is a $\langle\cdot,\cdot\rangle$-orthogonal basis of $\mathfrak{g}$.

The components r_4, r_5 and the off diagonal element r_0 of the Ricci tensor for the center $\mathfrak{h}_0$ of the left-invariant metric corresponding to the inner products (9) are given as follows:

$$
\begin{aligned}
r_4 &= \frac{u_4}{4}\left(\frac{1}{{x_{12}}^2}\left\{\begin{matrix}4\\(12)(12)\end{matrix}\right\} + \frac{1}{{x_{13}}^2}\left\{\begin{matrix}4\\(13)(13)\end{matrix}\right\} + \frac{1}{{x_{23}}^2}\left\{\begin{matrix}4\\(23)(23)\end{matrix}\right\}\right)\\
r_5 &= \frac{u_5}{4}\left(\frac{1}{{x_{12}}^2}\left\{\begin{matrix}5\\(12)(12)\end{matrix}\right\} + \frac{1}{{x_{13}}^2}\left\{\begin{matrix}5\\(13)(13)\end{matrix}\right\} + \frac{1}{{x_{23}}^2}\left\{\begin{matrix}5\\(23)(23)\end{matrix}\right\}\right)\\
r_0 &= \frac{\sqrt{u_4u_5}}{4}\left\{\frac{bd}{{x_{12}}^2}\frac{(\ell+m)}{(\ell+m+n)}\right.\\
&\quad + \frac{1}{{x_{13}}^2(\ell+m)}\left(\ell ac + \frac{\sqrt{\ell mn}}{\sqrt{(\ell+m+n)}}(ad+cb) + \frac{bdmn}{(\ell+m+n)}\right)\\
&\quad \left. + \frac{1}{{x_{23}}^2(\ell+m)}\left(mac - \frac{\sqrt{\ell mn}}{\sqrt{(\ell+m+n)}}(ad+cb) + \frac{bdn\ell}{(\ell+m+n)}\right)\right\}.
\end{aligned}
$$

The components $r_1, r_2, r_3, r_{12}, r_{13}, r_{23}$ of the left-invariant metric corresponding to the inner products (9) are given as follows:

$$
\begin{aligned}
r_1 &= \frac{\ell}{4N}\frac{1}{u_1} + \frac{u_1}{4N}\left(\frac{m}{{x_{12}}^2} + \frac{n}{{x_{13}}^2}\right), \\
r_2 &= \frac{m}{4N}\frac{1}{u_2} + \frac{u_2}{4N}\left(\frac{\ell}{{x_{12}}^2} + \frac{n}{{x_{23}}^2}\right), \\
r_3 &= \frac{n}{4N}\frac{1}{u_3} + \frac{u_3}{4N}\left(\frac{\ell}{{x_{13}}^2} + \frac{m}{{x_{23}}^2}\right), \\
r_{12} &= \frac{1}{2x_{12}} + \frac{n}{4N}\left(\frac{x_{12}}{x_{13}x_{23}} - \frac{x_{13}}{x_{12}x_{23}} - \frac{x_{23}}{x_{12}x_{13}}\right) - \frac{1}{4\ell mN}\frac{1}{{x_{12}}^2} \\
&\quad \times\Big((\ell^2-1)mu_1 + (m^2-1)\ell u_2 + (\ell+m)b^2u_4 + (\ell+m)d^2u_5\Big),
\end{aligned}
$$

Ricci tensor for $\mathrm{SU}(l+m+n)$

$$
\begin{aligned}
r_{13} &= \frac{1}{2x_{13}} + \frac{m}{4N}\left(\frac{x_{13}}{x_{12}x_{23}} - \frac{x_{12}}{x_{13}x_{23}} - \frac{x_{23}}{x_{12}x_{13}}\right) - \frac{1}{4\ell nN}\frac{1}{{x_{13}}^2} \\
&\times\Bigg((\ell^2-1)nu_1 + (n^2-1)\ell u_3 + \left(\frac{a^2\ell N}{\ell+m} + \frac{b^2mn}{\ell+m} + \frac{2ab\sqrt{\ell mn}\sqrt{N}}{\ell+m}\right)u_4 \\
&+ \left(\frac{c^2\ell N}{\ell+m} + \frac{d^2mn}{\ell+m} + \frac{2cd\sqrt{\ell mn}\sqrt{N}}{\ell+m}\right)u_5\Bigg), \\
r_{23} &= \frac{1}{2x_{23}} + \frac{\ell}{4N}\left(\frac{x_{23}}{x_{12}x_{13}} - \frac{x_{13}}{x_{12}x_{23}} - \frac{x_{12}}{x_{13}x_{23}}\right) - \frac{1}{4mnN}\frac{1}{{x_{23}}^2} \\
&\times\Bigg((m^2-1)nu_2 + (n^2-1)mu_3 + \left(\frac{a^2mN}{\ell+m} + \frac{b^2\ell n}{\ell+m} - \frac{2ab\sqrt{\ell mn}\sqrt{N}}{\ell+m}\right)u_4 \\
&+ \left(\frac{c^2mN}{\ell+m} + \frac{d^2\ell n}{\ell+m} - \frac{2cd\sqrt{\ell mn}\sqrt{N}}{\ell+m}\right)u_5\Bigg).
\end{aligned}
$$

To find non-naturally reductive Einstein metrics on $\mathrm{SU}(N)$ we need to solve the system:

$$\begin{aligned} & r_0 = 0 \\ & r_1 - r_2 = 0,\ r_2 - r_3 = 0,\ r_3 - r_4 = 0,\ r_4 - r_5 = 0, \\ & r_5 - r_{12} = 0,\ r_{12} - r_{13} = 0\ r_{13} - r_{23} = 0. \end{aligned} \tag{11}$$

The above system has 7 parameters a, b, c, d and ℓ, m, n.
In order to study its solutions, we make the simplification

$$a = d = 1,\ b = 0,\ \text{and}\ \ell = 1,\ m = 2.$$

(Note that Mori considered the case $a = d = 1,\ b = 0,\ c = 0$. So we have $x_{23} = x_{13}$ for these cases.)

So on $\mathrm{SU}(3+n)$ the system (11) reduces to

$$\begin{aligned} & r_0 = 0 \\ & r_2 - r_3 = 0,\ r_3 - r_4 = 0,\ r_4 - r_5 = 0, \\ & r_5 - r_{12} = 0,\ r_{12} - r_{13} = 0\ r_{13} - r_{23} = 0. \end{aligned} \tag{12}$$

In this system there is no u_1 variable. Now from $r_0 = 0$ we obtain by setting $x_{13} = 1$ that

$$c = \frac{\sqrt{2n(3+n)}}{(3+n)} \frac{(1 - x_{23}^2)}{(2 + x_{23}^2)}.$$

We substitute c to the system (12). Then we observe that the equations

$$r_4 - r_5 = 0,\ r_5 - r_{12} = 0, r_{12} - r_{13} = 0\ r_{13} - r_{23} = 0,$$

are linear with respect to u_2, u_3, u_4 and u_5!

In fact, (for simplicity, we change the variables $x_{12} = x_1, x_{23} = x_2$)

$$n u_4 {x_1}^2 {x_2}^4 + 3 u_4 {x_1}^2 {x_2}^4 - 9 u_5 {x_2}^4 + 4 n u_4 {x_1}^2 {x_2}^2 + 12 u_4 {x_1}^2 {x_2}^2 - 9 n u_5 {x_1}^2 {x_2}^2$$
$$-18 u_5 {x_2}^2 + 4 n u_4 {x_1}^2 + 12 u_4 {x_1}^2 = 0,$$

$$2 n x_1 {x_2}^4 + 3 u_2 {x_2}^3 + 9 u_5 {x_2}^3 - 4 n x_1 {x_2}^3 - 12 x_1 {x_2}^3 - 2 n {x_1}^3 {x_2}^2 + 6 n x_1 {x_2}^2$$
$$+6 n u_5 {x_1}^2 x_2 + 6 u_2 x_2 + 18 u_5 x_2 - 8 n x_1 x_2 - 24 x_1 x_2 - 4 n {x_1}^3 + 4 n x_1 = 0,$$

$$
\begin{aligned}
&-6n^2x_1x_2{}^7 - 6nx_1x_2{}^7 - 9nu_2x_2{}^6 - 9nu_5x_2{}^6 + 12n^2x_1x_2{}^6 + 36nx_1x_2{}^6\\
&+6n^2x_1{}^3x_2{}^5 + 6nx_1{}^3x_2{}^5 - 12n^2x_1{}^2x_2{}^5 - 36nx_1{}^2x_2{}^5 - 30n^2x_1x_2{}^5 - 18nx_1x_2{}^5\\
&+9nu_2x_1{}^2x_2{}^4 + 6n^2u_3x_1{}^2x_2{}^4 - 6u_3x_1{}^2x_2{}^4 + 2nu_4x_1{}^2x_2{}^4 + 6u_4x_1{}^2x_2{}^4\\
&+9nu_5x_1{}^2x_2{}^4 - 36nu_2x_2{}^4 - 36nu_5x_2{}^4 + 48n^2x_1x_2{}^4 + 144nx_1x_2{}^4 + 24n^2x_1{}^3x_2{}^3\\
&+24nx_1{}^3x_2{}^3 - 48n^2x_1{}^2x_2{}^3 - 144nx_1{}^2x_2{}^3 - 48n^2x_1x_2{}^3 + 36nu_2x_1{}^2x_2{}^2\\
&+24n^2u_3x_1{}^2x_2{}^2 - 24u_3x_1{}^2x_2{}^2 + 8nu_4x_1{}^2x_2{}^2 + 24u_4x_1{}^2x_2{}^2 - 36nu_2x_2{}^2\\
&-36nu_5x_2{}^2 + 48n^2x_1x_2{}^2 + 144nx_1x_2{}^2 + 24n^2x_1{}^3x_2 + 24nx_1{}^3x_2 - 48n^2x_1{}^2x_2\\
&-144nx_1{}^2x_2 - 24n^2x_1x_2 + 24nx_1x_2 + 36nu_2x_1{}^2 + 24n^2u_3x_1{}^2 - 24u_3x_1{}^2\\
&+8nu_4x_1{}^2 + 24u_4x_1{}^2 = 0,
\end{aligned}
$$

$$
\begin{aligned}
&18nx_2{}^7 - 12n^2x_1x_2{}^6 - 36nx_1x_2{}^6 + 6n^2u_3x_1x_2{}^6 - 6u_3x_1x_2{}^6 + 2nu_4x_1x_2{}^6\\
&+6u_4x_1x_2{}^6 + 6nx_1{}^2x_2{}^5 + 54nx_2{}^5 + 12n^2x_1x_2{}^5 + 36nx_1x_2{}^5 - 48n^2x_1x_2{}^4\\
&-144nx_1x_2{}^4 - 9nu_2x_1x_2{}^4 + 18n^2u_3x_1x_2{}^4 - 18u_3x_1x_2{}^4 + 6nu_4x_1x_2{}^4 - 72nx_2\\
&+18u_4x_1x_2{}^4 - 9nu_5x_1x_2{}^4 + 24nx_1{}^2x_2{}^3 + 48n^2x_1x_2{}^3 + 144nx_1x_2{}^3 + 144nx_1x_2\\
&-48n^2x_1x_2{}^2 - 144nx_1x_2{}^2 - 36nu_2x_1x_2{}^2 + 36nu_5x_1x_2{}^2 + 24nx_1{}^2x_2\\
&+48n^2x_1x_2 - 36nu_2x_1 - 24n^2u_3x_1 + 24u_3x_1 - 8nu_4x_1 - 24u_4x_1 = 0.
\end{aligned}
$$

By solving the equations with respect to u_2, u_3, u_4 and u_5, we see that these u_2, u_3, u_4 and u_5 can be expressed as rational functions of x_1, x_2. We substitute the u_2, u_3, u_4 and u_5 into the equations

$$r_2 - r_3 = 0, \quad r_3 - r_4 = 0.$$

Then we obtain two polynomials $F_1(x_1, x_2)$ and $F_2(x_1, x_2)$ with parameter n.
We can see that the system of equations $\{F_1(x_1, x_2) = 0, F_2(x_1, x_2) = 0\}$ reduces to solve polynomials of x_1 and x_2 with degree 58 (for $n = 2$ degree 50) by using Gröbner basis or taking resultant.

For $n = 2$, that is, on $\mathrm{SU}(5)$, by solving the equations $F_1(x_1, x_2) = 0$ and $F_2(x_1, x_2) = 0$ and substitute into u_2, u_3, u_4 and u_5, we obtain two solutions which correspond to non-naturally reductive Einstein metrics which are given approximately:

$$\{x_{12} \approx 0.52971824,\ x_{13} = 1,\ x_{23} \approx 0.96176370,\ u_2 \approx 0.41848636,\\ u_3 \approx 0.32539315,\ u_4 \approx 1.3614688,\ u_5 \approx 0.56310003\},$$

$$\{x_{12} \approx 1.8877961,\ x_{13} = 1,\ x_{23} \approx 1.81561371, u_2 \approx 0.61427591,\\ u_3 \approx 0.79001690, u_4 \approx 1.4193906, u_5 \approx 1.9248703\}.$$

For $n = 3$, that is, on $\mathrm{SU}(6)$, by solving the equations $F_1(x_1, x_2) = 0$ and $F_2(x_1, x_2) = 0$ and substitute into u_2, u_3, u_4 and u_5, we obtain two solutions which correspond to non-naturally reductive Einstein metrics which are given approximately:

$$\{x_{12} \approx 41154990,\ x_{13} = 1,\ x_{23} \approx 0.97927089, u_2 \approx 0.32689357, u_3 \approx 0.37979646, u_4 \approx 1.4193906, u_5 \approx 0.43706645\}.$$

$$\{x_{12} \approx 1.7673758,\ x_{13} = 1,\ x_{23} \approx 1.6924525,\ u_2 \approx 0.46014821, u_3 \approx 0.84887565,\ u_4 \approx 1.46495989,\ u_5 \approx 1.7703211\},$$

- Suppose that a homogeneous space G/H has the following property: the modules $\mathfrak{p}$ is decomposed as a direct sum of three Ad(H)-invariant irreducible modules pairwise orthogonal with respect to B (negative of Killing form), that is,

$$\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3$$

such that

$$[\mathfrak{p}_i, \mathfrak{p}_i] \subset \mathfrak{h} \quad \text{for} \;\; i \in \{1, 2, 3\}.$$

Homogeneous spaces with this property are called **generalized Wallach spaces** (due to Yu. Nikonorov).

- Note that the inclusion $[\mathfrak{p}_i, \mathfrak{p}_i] \subset \mathfrak{h}$ implies that $\mathfrak{k}_i = \mathfrak{h} \oplus \mathfrak{p}_i$ is a subalgebra of $\mathfrak{g}$ for any i, and the pair $(\mathfrak{k}_i, \mathfrak{h})$ is irreducible symmetric (it could be non-effective). We also see that

$$[\mathfrak{p}_i, \mathfrak{p}_j] \subset \mathfrak{p}_k$$

for distinct i, j, k. Therefore,

$$[\mathfrak{p}_i \oplus \mathfrak{p}_k, \mathfrak{p}_j \oplus \mathfrak{p}_k] \subset \mathfrak{h} \oplus \mathfrak{p}_i, \quad \{i, j, k\} = \{1, 2, 3\},$$

and all the pairs $(\mathfrak{g}_i, \mathfrak{k}_i)$ are also irreducible symmetric.

Examples of generalized Wallach spaces

- Wallach spaces:

$$\mathrm{SU}(3)/T^2, \ \mathrm{Sp}(3)/(\mathrm{Sp}(1)\times\mathrm{Sp}(1)\times\mathrm{Sp}(1)), \ F_4/\,\mathrm{Spin}(8).$$

  These spaces are also interesting in that they admit invariant Riemannian metrics of positive sectional curvature.

- Other examples of generalized Wallach spaces are some generalized flag manifolds such as

$$\mathrm{SU}(n_1+n_2+n_3)/S\,(\mathrm{U}(n_1)\times\mathrm{U}(n_2)\times\mathrm{U}(n_3)),$$

$$\mathrm{SO}(2n)/(\mathrm{U}(1)\times\mathrm{U}(n-1)), \quad E_6/(\mathrm{U}(1)\times\mathrm{U}(1)\times\mathrm{Spin}(8))$$

  There are two more 3-parameter families of generalized Wallach spaces:

$$\mathrm{SO}(n_1+n_2+n_3)/(\mathrm{SO}(n_1)\times\mathrm{SO}(n_2)\times\mathrm{SO}(n_3)),$$

$$\mathrm{Sp}(n_1+n_2+n_3)/(\mathrm{Sp}(n_1)\times\mathrm{Sp}(n_2)\times\mathrm{Sp}(n_3))$$

Generalized Wallach spaces

- Yurii Nikonorov has classified all generalized Wallach spaces for compact simple Lie groups. (Geometriae Dedicata, 2016, DOI 10.1007/s10711-015-0119-z and/or in ArXiv: 1411.3131v1 12 Nov 2014)

  There are 15 cases with 5 series for classical groups and 10 exceptional Lie groups.